Op-Ex—An Optimal-Explicit Guidance Algorithm for Powered Flight outside the Atmosphere

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The principal difficulties in applying optimal control theory to the two-point boundary-value problem which describes rocket powered flight outside the atmosphere are in integrating the state and costate equations fast and accurately, and in re-estimating initial values of the costate to achieve rapid convergence. This paper describes an algorithm called Op-Ex which applies Newton's method in a standard fashion to satisfy the second difficulty, and employs an integration technique not unlike those used in current "explicit guidance" algorithms to satisfy the first. Op-Ex has flexible provisions for propulsion performance description and allows an unlimited number of rocket stages to be specified. For any trajectory phase, any of a variety of alternative terminal requirements can be easily specified, and for each, Op-Ex will generate an optimal (fuel-minimal) trajectory. Op-Ex is computationally efficient, to be practical for real-time guidance, yet is sufficiently accurate to be used for rapid prediction of fuel requirements for flight planning purposes.

Introduction

It is becoming increasingly apparent that the guidance techniques currently being used, because they are tailored to a restricted class of guidance problems, will not be able to meet the diversity of future guidance requirements as the frequency and variety of space missions increase. For this reason a versatile guidance algorithm called Op-Ex has been developed which is suitable for exoatmospheric ascent, orbit transfers, direct rendezvous, direct intercept, de-orbit—in fact, for all phases of powered flight outside the atmosphere except possibly the terminal phase of rendezvous. For each of these guidance requirements Op-Ex produces a trajectory that is optimal in the sense that fuel expenditure is minimal.

A multiplicity of alternative terminal conditions can be easily specified for any mission phase. All that is required is the specification of six final condition equations (according to a set of rules which will be described) which completely define the mission type. These equations, when their values simultaneously vanish, determine the unique optimal trajectory.

Op-Ex can optimally perform any magnitude of plane change within the rocket's capabilities during ascent or during any orbital maneuver. Provisions for the propulsion performance description are such that, in theory, any modelable function of time can be used to represent engine mass-flow rate or exhaust velocity in any rocket stage, and an unlimited number of stages can be specified.

Op-Ex is conceptually similar to a guidance scheme described by Brown and Johnson,[1] but differs substantially in computational techniques.

Notation

The equations presented are written in vector-matrix form. Vectors appear as lower-case symbols in bold face type. Matrices are always designated by upper-case symbols and scalars by lower-case symbols. Differentiation with respect to time is designated by a dot above the variable. The symbol definitions are included in the text.

Statement of the Problem

The equations of state for rocket-powered flight outside the atmosphere are

$$\ddot{\mathbf{r}} = \dot{\mathbf{v}} = \mathbf{g}(\mathbf{r},t) + a(t)\mathbf{u}(t) \tag{1}$$

where $\mathbf{r}$ is position, $\mathbf{v}$ is velocity, $\mathbf{g}$ is acceleration due to gravity, $a(t)$ is the magnitude of thrust acceleration, and $\mathbf{u}(t)$ is a unit vector giving the direction of thrust acceleration. For an n-stage rocket, equations defining $a(t)$ are

$$\left.\begin{aligned} a(t) &= \tau_j/m_j \\ \dot{m}_j &= -\tau_j/c_j^* \end{aligned}\right\} \; j = 1,2, \ldots, n \tag{2}$$

where m is vehicle mass, τ is thrust, and c^* is the effective exhaust velocity. c^*, τ, and m may change discontinuously at staging times. Also, c^* and τ may be functions of the burning time since state ignition. For solid fuel rockets this is usually the case.

In many cases, the control of thrust magnitude is prescribed or trivial (full on until cutoff, zero thereafter) so only the optimization of steering need be considered. For such cases, the application of Pontryagin's maximum principle reduces the problem of trajectory optimization to a two-point boundary-value problem, and yields the costate equations

$$\ddot{\boldsymbol{\lambda}} = G\boldsymbol{\lambda} \tag{3}$$

$$\mathbf{u} = \boldsymbol{\lambda}/|\boldsymbol{\lambda}| \tag{4}$$

where G is the gravity gradient matrix and $\boldsymbol{\lambda}$ is the "primer vector." For an inverse-square central field, $\mathbf{g}$ and G are given by

$$\mathbf{g} = -(\mu/|\mathbf{r}|^3)\mathbf{r} \tag{5}$$

$$G = -(\mu/|\mathbf{r}|^3)[I - (3/|\mathbf{r}|^2)\mathbf{r}\mathbf{r}^T] \tag{6}$$

where I is the identity matrix, $\mathbf{r}^T$ is the transpose of the column vector $\mathbf{r}$, and μ is the coefficient of gravitational mass attraction.

The initial conditions of the two-point boundary-value problem generally consist of given values of $\mathbf{r}_0$, $\mathbf{v}_0$, m_0, and

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t_0. For cases in which the final mass is to be maximized, the final conditions consist of $k \leq 6$ mission conditions (prescribed relations among the final values of $\mathbf{r}_f$, $\mathbf{v}_f$, and t_f) plus 6-k transversality conditions which come from optimal control theory. The transversality conditions can be analytically derived from the mission conditions by the requirement that

$$\dot{\boldsymbol{\lambda}} \cdot \delta\mathbf{r} - \boldsymbol{\lambda} \cdot \delta\mathbf{v} = 0 \qquad t \geq t_f \tag{7}$$

must hold for every pair of infinitesimal variations $\delta\mathbf{v}$, $\delta\mathbf{r}$ consistent with mission conditions, and at every point on the subsequent unpowered trajectory.

A more complete justification of the equations presented here can be found in Ref. 2.

Numerical Solution of the Optimal Problem

Numerical integration of Eqs. (1–6) from some initial time t_0 to an estimated final time t_f, starting from the initial states $\mathbf{r}_0$ and $\mathbf{v}_0$ and estimated initial costates $\boldsymbol{\lambda}_0$ and $\dot{\boldsymbol{\lambda}}_0$, determines the final states $\mathbf{r}_f$ and $\mathbf{v}_f$ and the final costates $\boldsymbol{\lambda}_f$ and $\dot{\boldsymbol{\lambda}}_f$. Seven final conditions derived from k specified final state conditions and 6-k transversality conditions, and a desired normalization of $\boldsymbol{\lambda}_0$, must be brought to null by iterative adjustment of the seven parameters $\boldsymbol{\lambda}_0$, $\dot{\boldsymbol{\lambda}}_0$, and t_f. To this end a Newton-Raphson iteration procedure will be used.

If the same trajectory integration algorithm is to be used for prelaunch (or preignition) flight planning of all powered phases and for real-time guidance during the powered phases, the integration must be both fast and accurate. An obvious method of increasing the speed of solution is to increase the step-size used in numerical integration. Since the guidance is "closed loop" and can be made accurate near cutoff at small cost, this does not cause guidance errors. However, it has three possible adverse effects. 1) The steering policy becomes nonoptimal, incurring a fuel penalty. 2) The initial estimate of t_f becomes inaccurate, which distorts flight planning. 3) Truncation errors in the computation of sensitivity coefficients (which are required for iterative correction of the estimated parameters $\boldsymbol{\lambda}_0$, $\dot{\boldsymbol{\lambda}}_0$, and t_f) cause them to differ from the actual sensitivities of the numerically integrated equations. This degrades the convergence of the iterative solution.

However, it is known that present day "explicit guidance" schemes, which are essentially one-step integrations of the trajectory equations improved by closed formulas for certain thrust integrals, have small performance penalties. It may therefore be expected that an "explicit-predictor" integration scheme which resembles the application of explicit guidance to each integration step will have negligible performance penalty for very large integration steps. This, in fact, is the case. It has been demonstrated that one integration step per stage of a typical present day booster produces a performance penalty of less than 2.0 fps.

Errors in initial estimates of t_f, made when the time remaining until thrust cutoff is large, are quite significant for conventional explicit guidance schemes, but have been found to be acceptably small when the explicit-predictor integration scheme is used with large steps. It has been demonstrated, using a typical booster, that integration steps of 100 sec produce a cutoff prediction error of less than 0.7 sec in cases where the time remaining until cutoff is greater than 700 sec. Even larger time steps are believed usable for flight planning. The prediction error does not compromise guidance accuracy since it diminishes rapidly with decreasing time-until-cutoff.

The remaining factor governing usable stepsize is the effect of truncation errors on the convergence of the iterative solution. This effect disappears if the sensitivity coefficients are computed by finite differences. This is done by first generating an "unperturbed" trajectory and evaluating the final conditions. Then "perturbed" trajectories are generated by changing components of the initial costate vector one at a time by small specified increments. Subtracting the final conditions of each perturbed trajectory from the corresponding final conditions of the unperturbed trajectory, and dividing by the change in the initial costate component, gives one column of the sensitivity matrix.

Computing sensitivities by finite difference is similar to performing experiments on the numerical trajectory solution. Regardless of truncation errors, approximations in the equations, etc., the sensitivity coefficients generated will give an accurate prediction (except for roundoff and nonlinearity effects) of the way in which the numerical trajectory solution will change if its parameters are changed.

Experience with the ascent care has shown that there is no difficulty in choosing costate component increments large enough to avoid trouble from roundoff errors, but small enough for linearity to hold. In trial experiments, the sizes of the increments were varied more than three orders of magnitude without appreciable degradation of convergence.

Explicit-Predictor Integration Algorithm

The "explicit-predictor" integration algorithm is designed to take advantage of closed formula for integrals involving thrust acceleration. The formulas presented here assume that rocket engine mass flow rate $\dot{m}$ and effective exhaust velocity c^* are constant. This should not be construed as a limitation of the technique. In theory, at least, any modelable function of burning time can be used for either $\dot{m}$ or c^*. For example, a model in which the level of thrust is abruptly altered sometime after engine ignition could be implemented within the framework of the algorithm.

The equations of the integration algorithm will be shown for the jth rocket stage and the nth integration time step. Since engine parameters and mass are generally discontinuous at staging, the integration time step is selected so that no more than one stage is ever included in any integral evaluation. In other words

$$\Delta t_n = \mathrm{Min}(\Delta t_{\max}, t_j - t_n), \qquad t_{n+1} = t_n + \Delta t_n \tag{8}$$

where t_j is the time at the end of state j and $\Delta t_{\max}$ is the largest time step allowed.

The primer vector and the steering vector are predicted at t_m and t_{n+1} by using the second-order equations

$$\begin{aligned} \boldsymbol{\lambda}_m &= \boldsymbol{\lambda}_n + \dot{\boldsymbol{\lambda}}_n \Delta t_n/2 + \ddot{\boldsymbol{\lambda}}_n \Delta t_n^2/8, \qquad \mathbf{u}_m = \boldsymbol{\lambda}_m/|\boldsymbol{\lambda}_m| \\ \boldsymbol{\lambda}_{n+1} &= \boldsymbol{\lambda}_n + \dot{\boldsymbol{\lambda}}_n \Delta t_n + \ddot{\boldsymbol{\lambda}}_n \Delta t_n^2/2, \qquad \mathbf{u}_{n+1} = \boldsymbol{\lambda}_{n+1}/|\boldsymbol{\lambda}_{n+1}| \end{aligned} \tag{9}$$

and

$$t_m = t_n + \Delta t_n/2$$

The thrust acceleration term $a(t)\mathbf{u}(t)$ is separated into two parts, that which defines its magnitude, $a(t)$, and that which defines its direction, $\mathbf{u}(t)$. Over each integration step, $\mathbf{u}(t)$ is approximated by a second-order vector function of time

$$\mathbf{u}(t) = \mathbf{u}_m + \boldsymbol{\alpha}_n\{t - t_m\} + \boldsymbol{\beta}_n\{t - t_m\}^2 \tag{10}$$

$$t_n \leq t \leq t_{n+1}$$

Given the three values of $\mathbf{u}$, the vector coefficients are

$$\begin{aligned} \boldsymbol{\alpha}_n &= [\mathbf{u}_{n+1} - \mathbf{u}_n]/\Delta t_n \\ \boldsymbol{\beta}_n &= 2[\mathbf{u}_{n+1} + \mathbf{u}_n - 2\mathbf{u}_m]/\Delta t_n^2 \end{aligned} \tag{11}$$

To complete the single and double integrals of thrust acceleration from t_n to t_{n+1}, it is necessary to evaluate six integrals

$$\zeta_{i+1} = \int_{t_n}^{t_{n+1}} \frac{c_j^*(\sigma - t_m)^i}{\nu_j + t_{j-1} - \sigma} d\sigma$$
$$i = 0,1,2 \quad (12)$$
$$\zeta_{i+4} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{\eta} \frac{c_j^*(\sigma - t_m)^i}{\nu_j + t_{j-1} - \sigma} d\sigma d\eta$$

where $\nu_j = -m_{0j}/\dot{m}_j$; m_{0j} is the mass at the ignition of stage j. For the assumptions just made, namely that c_j^* and $\dot{m}_j$ are constant

$$\zeta_1 = c_j^* \log_e(1 + \rho) \quad (13a)$$

$$\zeta_4 = c_j^* \Delta t_n - (\nu_j + t_{j-1} - t_{n+1})\zeta_1 \quad (13b)$$

$$\zeta_2 = -\zeta_4 - (t_m - t_{n+1})\zeta_1 \quad (13c)$$

$$\zeta_3 = (\nu_j + t_{j-1} - t_m)(\zeta_2 - c_j^* \Delta t_n) + c_j^*(\nu_j + t_{j-1} - t_{n+1})^2(2\rho + \rho^2)/2 \quad (13d)$$

$$\zeta_5 = (\nu_j + t_{j-1} - t_m)\zeta_4 - c_j^* \Delta t_n^2/2 \quad (13e)$$

$$\zeta_6 = (\nu_j + t_{j-1} - t_m)(\zeta_5 - c_j^* \Delta t_n^2/2) - c_j^*[(\nu_j + t_{j-1} - t_{n+1})^3(3\rho + 3\rho^2 + \rho^3)/6 - (\nu_j + t_{j-1} - t_n)^2 \Delta t_n/2] \quad (13f)$$

where

$$\rho = \Delta t_n/(\nu_j + t_{j-1} - t_{n+1})$$

The thrust acceleration contributions to position and velocity are

$$\begin{aligned} \Delta \mathbf{r}_{n+1}' &= \Delta \mathbf{r}_n' + \Delta v_n' \Delta t_n + \zeta_4 \mathbf{u}_m + \zeta_5 \boldsymbol{\alpha}_n + \zeta_6 \boldsymbol{\beta}_n \\ \Delta \mathbf{v}_{n+1}' &= \Delta \mathbf{v}_n' + \zeta_1 \mathbf{u}_m + \zeta_2 \boldsymbol{\alpha}_n + \zeta_3 \boldsymbol{\beta}_n \end{aligned} \quad (14)$$

The gravitational acceleration contributions to position are

$$\Delta \mathbf{r}_{n+1}'' = \Delta \mathbf{r}_n'' + \Delta \mathbf{v}_n'' \Delta t_n + \mathbf{g}_n \Delta t_n^2/2 + G_n \mathbf{v}_n \Delta t_n^3/6 \quad (15)$$

Total position is

$$\mathbf{r}_{n+1} = \mathbf{r}_0 + \mathbf{v}_0(t_{n+1} - t_0) + \Delta \mathbf{r}_{n+1}' + \Delta \mathbf{r}_{n+1}'' \quad (16)$$

Gravitational acceleration and the gravity-gradient matrix can now be evaluated using the predicted value of $\mathbf{r}_{n+1}$ as specified by Eqs. (5) and (6); then $\ddot{\boldsymbol{\lambda}}_{n+1}$ can be evaluated

$$\ddot{\boldsymbol{\lambda}}_{n+1} = G_{n+1} \boldsymbol{\lambda}_{n+1} \quad (17)$$

Finally, velocity and primer vector rate are

$$\begin{aligned} \Delta \mathbf{v}_{n+1}'' &= \Delta \mathbf{v}_n'' + (\mathbf{g}_{n+1} + \mathbf{g}_n) \Delta t_n/2 \\ \mathbf{v}_{n+1} &= \mathbf{v}_0 + \Delta \mathbf{v}_{n+1}' + \Delta \mathbf{v}_{n+1}'' \\ \dot{\boldsymbol{\lambda}}_{n+1} &= \dot{\boldsymbol{\lambda}}_n + (\ddot{\boldsymbol{\lambda}}_{n+1} + \ddot{\boldsymbol{\lambda}}_n) \Delta t_n/2 \end{aligned} \quad (18)$$

Equations (8–18) are the explicit-predictor integration technique. As integration proceeds, the index j in Eqs. (13) must refer to the "current" stage. In a real-time guidance application, $\mathbf{r}_0$ and $\mathbf{v}_0$ would be determined from the onboard navigation system.

Iterative Determination of Estimated Parameters

As pointed out previously, seven final conditions must be satisfied by iteratively adjusting the estimated parameters $\boldsymbol{\lambda}_0$, $\dot{\boldsymbol{\lambda}}_0$, and t_f. But, as can be deduced from Eq. (4), the magnitude of the primer vector at t_0 is irrelevant. Only its direction is important, and its direction can be completely specified by two parameters. By an expeditious choice of coordinates a "preferred coordinate" can be established along which the component of $\boldsymbol{\lambda}_0$ must be finite and positive; this limits the selection to a hemisphere in three-dimensional space. This component can then be left invariant and the other two components adjusted relative to it, thus reducing the number of iteratively corrected parameters from seven to six. An excellent and convenient choice for the "preferred coordinate" is the direction of the vector difference between the initial and estimated final velocities.

To iteratively adjust the estimated parameters, it is necessary to establish the sixth-order matrix of sensitivities, the Jacobian matrix, which relates the variations of estimated parameters to the variations in final conditions. The Jacobian matrix is computed by employing finite-difference techniques. Only five "perturbed" and one "unperturbed" trajectories must be generated. (At this point, one can see the obvious advantage of an extremely rapid trajectory generation technique.) Each perturbed trajectory yields one column of the Jacobian matrix. The sixth column, which represents the changes induced in the final conditions by a unit charge of the final time, can be evaluated by extrapolating the final state of the unperturbed trajectory over some small time increment and then re-evaluating the final conditions.

If the final conditions for the six perturbed trajectories are designated by column vectors $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_6$ of dimension 6, and the unperturbed trajectory final conditions by $\boldsymbol{\varepsilon}_0$, the Jacobian matrix is

$$\begin{aligned} J = [&(\boldsymbol{\varepsilon}_0 - \boldsymbol{\varepsilon}_1)/\Delta\dot{\lambda}, (\boldsymbol{\varepsilon}_0 - \boldsymbol{\varepsilon}_2)/\Delta\dot{\lambda}, (\boldsymbol{\varepsilon}_0 - \boldsymbol{\varepsilon}_3)/\Delta\dot{\lambda}, \\ &(\boldsymbol{\varepsilon}_0 - \boldsymbol{\varepsilon}_4)/\Delta\lambda, (\boldsymbol{\varepsilon}_0 - \boldsymbol{\varepsilon}_5)/\Delta\lambda, (\boldsymbol{\varepsilon}_0 - \boldsymbol{\varepsilon}_6)/\Delta t_f] \end{aligned} \quad (19)$$

The Newton-Raphson iteration equation in vector form is

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \\ \lambda_2 \\ \lambda_3 \\ t_f \end{bmatrix} \leftarrow \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \\ \lambda_2 \\ \lambda_3 \\ t_f \end{bmatrix} + J^{-1}(\gamma \boldsymbol{\varepsilon}_0) \quad (20)$$

where the subscripts of λ and $\dot{\lambda}$ refer to the first, second, and third components of $\boldsymbol{\lambda}_0$ and $\dot{\boldsymbol{\lambda}}_0$. Note that λ_1 is arbitrarily chosen to be along the "preferred" coordinate.

In Eq. (20), γ is a convergence progress control parameter which is adjusted according to a "performance indicator." The performance indicator is defined as

$$\sum_{i=1}^{6} \omega_i \varepsilon_{0i}^2 \quad (21)$$

which is the weighted sum square of the unperturbed final conditions. The weighting factors ω_i are required to maintain dimensional compatibility among the final conditions.

If the Jacobian matrix is correct and nonsingular, progress must be made by Eq. (20) in reducing the final conditions if a sufficiently small γ is used. On the other hand, if γ is too small the reduction process will be unnecessarily sluggish. The rule used to adaptively adjust γ is as follows: if the performance indicator on the present iteration is less than it was on the last iteration, γ is set

$$\gamma \leftarrow \text{Min}(2\gamma, 1)$$

and the evaluation of a new Jacobian proceeds. If the performance indicator is greater than it was on the last iteration, γ is set

$$\gamma \leftarrow \gamma/2$$

and Eq. (20) is retried with the same Jacobian. Termination of iteration occurs when the performance indicator falls below some specified minimum.

Specification of Final Conditions

It is necessary to specify six final conditions which uniquely define a mission type and, when caused to vanish, satisfy the mission objectives. Of the six, k are specified final state conditions (or functions of final state conditions) and 6-k are transversality conditions which insure an optimal trajectory.

Three basic guidance options will be described below. Other options can be derived in a straight-forward manner.

1. Injection into Orbit, with Specification of Orbit Size, Shape, and Orientation

For injection into orbit with prescribed orbit size, shape, and orientation, the only degrees of freedom remaining are the time and location at which orbit insertion is to occur. If the desired orbit is specified by its radius and velocity vectors at perigee, $\mathbf{r}_p$ and $\mathbf{v}_p$, the required velocity vector at any location in the orbit is given by

$$\mathbf{v}_r = -\{(\mu/s)^{1/2}\sin\theta\}\mathbf{r}_p/|\mathbf{r}_p| + \{1 - (1-\cos\theta)|\mathbf{r}_p|/s\}\mathbf{v}_p \quad (22)$$

where θ is the true anomaly. The semilatus rectum s is given by

$$s = (\mathbf{v}_p\cdot\mathbf{v}_p)(\mathbf{r}_p\cdot\mathbf{r}_p)/\mu \quad (23)$$

The magnitude of the required radius at any point in the orbit is

$$r_r = s/(1 + e\cos\theta) \quad (24)$$

where the orbit eccentricity is given by

$$e = (s/|\mathbf{r}_p| - 1)^{1/2} \quad (25)$$

The unit vector normal to the orbital plane is

$$\mathbf{n} = \mathbf{r}_p \times \mathbf{v}_p/|\mathbf{r}_p \times \mathbf{v}_p| \quad (26)$$

As just shown, the required velocity vector and required radius are functions of the true anomaly. The true anomaly of the generated trajectory is given by

$$\theta = \tan^{-1}\{(\mathbf{r}_f\cdot\mathbf{v}_p/|\mathbf{v}_p|)/(\mathbf{r}_f\cdot\mathbf{r}_p/|\mathbf{r}_p|)\} \quad (27)$$

Five of the six required final conditions are

$$\epsilon_{n2} = r_r(\theta) - |\mathbf{r}_f|$$
$$\epsilon_{n3} = -\mathbf{n}\cdot\mathbf{r}_f \quad (28)$$
$$\begin{bmatrix}\epsilon_{n4}\\ \epsilon_{n5}\\ \epsilon_{n6}\end{bmatrix} = \mathbf{v}_r(\theta) - \mathbf{v}_f$$

These five quantities are sufficient to satisfy all mission specifications. The sixth quantity gives the deviation from satisfaction of a transversality condition, which is

$$\epsilon_{n1} = \boldsymbol{\lambda}_f\cdot\mathbf{g}_f - \dot{\boldsymbol{\lambda}}_f\cdot\mathbf{v}_f \quad (29)$$

and expresses the fact that phase-in-orbit is not specified. This is the well-known transversality condition for "time-free" cases.

2. Direct Ascent to Rendezvous, with Rendezvous Time Specified

Direct ascent to rendezvous means that after thrust is initially terminated, only a velocity matching burn at the specified rendezvous radius is required. For this case, the required velocity $\mathbf{v}_r$ is determined by means of Lambert's theorem. The equations used to determine $\mathbf{v}_r$ can be found in Ref. 3 and are not repeated here. The remaining three final conditions must be specified by transversality conditions.

To determine the applicable transversality conditions, the equation of perturbed motions is obtained from Eq. (1) and is

$$\delta\ddot{\mathbf{r}} = \delta\dot{\mathbf{v}} = G\delta\mathbf{r} + \delta\mathbf{a} \quad (30)$$

It can be seen that on an unpowered trajectory, where $\delta\mathbf{a} = 0$, Eq. (3) is identical in form to Eq. (30). Hence, the well-known state transition matrix solution for Eq. (30) must also be a solution for Eq. (3). Symbolically,

$$\begin{bmatrix}\boldsymbol{\lambda}_i\\ \dot{\boldsymbol{\lambda}}_i\end{bmatrix} = \Phi(t_i,t_f)\begin{bmatrix}\boldsymbol{\lambda}_f\\ \dot{\boldsymbol{\lambda}}_f\end{bmatrix} \quad (31)$$

where t_i is the time of rendezvous. If $\dot{\boldsymbol{\lambda}}_i$ is assumed to be zero, i.e., a fixed attitude orbit matching burn, the required primer vector rate at thrust cutoff is given by

$$\dot{\boldsymbol{\lambda}}_r = \Phi_{12}^{-1}\{\boldsymbol{\lambda}_i - \Phi_{11}\boldsymbol{\lambda}_f\} \quad (32)$$

where

$$\Phi(t_i,t_f) = \begin{bmatrix}\Phi_{11} & \Phi_{12}\\ \Phi_{21} & \Phi_{22}\end{bmatrix}$$

All that remains is to determine $\boldsymbol{\lambda}_i$. From optimal control theory it can be shown that the magnitudes of the primer vector at the first thrust termination and at the beginning of the rendezvous burn must be equal. The direction of the primer vector at the beginning of the rendezvous burn must be approximately in the direction of the velocity-to-be-gained vector. Hence,

$$\boldsymbol{\lambda}_i \simeq |\boldsymbol{\lambda}_f|[\mathbf{v}_t - \mathbf{v}_i]/|\mathbf{v}_t - \mathbf{v}_i| \quad (33)$$

where $\mathbf{v}_t$ is target velocity and $\mathbf{v}_i$ is interceptor velocity at rendezvous. The three additional final conditions are

$$\begin{bmatrix}\epsilon_{n1}\\ \epsilon_{n2}\\ \epsilon_{n3}\end{bmatrix} = \Phi_{12}^{-1}\left\{\frac{|\boldsymbol{\lambda}_f|[\mathbf{v}_t - \mathbf{v}_i]}{|\mathbf{v}_t - \mathbf{v}_i|} - \Phi_{11}\boldsymbol{\lambda}_f\right\} - \dot{\boldsymbol{\lambda}}_f \quad (34)$$

Reference 4 describes an elegant way of computing the required state transition matrix which is valid for all orbits except the degenerate rectilinear orbit.

3. Direct Ascent to Intercept, with Intercept Time Specified

Direct ascent to intercept is very similar to direct ascent to rendezvous just described. The difference is that no velocity matching burn is performed at intercept. The required velocity for intercept is computed in the same manner as for rendezvous, but the transversality conditions are not.

Examining the point at intercept, the perturbation in position $\delta\mathbf{r}$ must, by definition, be zero. Since there is no constraint on velocity at intercept, the only way Eq. (7) can be satisfied is if $\boldsymbol{\lambda}$ at intercept is also zero. Hence, Eq. (34) can be rewritten for the intercept case and is

$$\begin{bmatrix}\epsilon_{n1}\\ \epsilon_{n2}\\ \epsilon_{n3}\end{bmatrix} = -\Phi_{12}^{-1}\Phi_{11}\boldsymbol{\lambda}_f - \dot{\boldsymbol{\lambda}}_f \quad (35)$$

Use of Op-Ex for Active Guidance

The use of Op-Ex for active guidance presupposes the existence of a means by which the velocity and position of the rocket can be measured. These measurements supply the initial state $\mathbf{r}_0$ and $\mathbf{v}_0$ from which forward trajectory integration can be started. The state and costate equations are then iteratively integrated until convergence is achieved and the parameters $\boldsymbol{\lambda}_0$, $\dot{\boldsymbol{\lambda}}_0$, and t_f are determined.

The predictive process is repeated at specified intervals as time progresses. In practice, repetition rates of once every 10 sec have been found to be more than adequate in early stages. Nearing thrust cutoff, the rate is increased to about once every one to five seconds so that the time of thrust cutoff is more frequently updated. Thrust cutoff is achieved by time countdown using the most recent estimate of t_f. During the intervals between guidance predictions the thrust axis of the rocket is steered using the extrapolated primer vector

$$\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}_0 + \dot{\boldsymbol{\lambda}}_0\{t - t_0\} + \ddot{\boldsymbol{\lambda}}_0\{t - t_0\}^2/2 \quad (36)$$

Table 1 Initial conditions for ascent trajectories

Radius	21113739 ft
Velocity magnitude	9460.213 fps
Velocity direction (measured from the geocentric radius)	69.30567°

It is often desirable to revert to a velocity-to-be-gained formulation when very close to cutoff. Otherwise, in the presence of off-nominal performance the guidance algorithm might call for very high attitude rates just before cutoff to correct trivial position errors. The algorithm formulated can easily be altered to a velocity-to-be gained algorithm for use in the last few seconds before cutoff. It is necessary for this discussion to assume that the last three final conditions are always the required velocity components.

The Jacobian matrix is partitioned into four third-order matrices.

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

Then Eq. (20) can be rewritten to be solved sequentially as

$$\delta\dot{\boldsymbol{\lambda}} = [J_{11} - J_{12}J_{22}^{-1}J_{21}]^{-1}\left\{\begin{bmatrix} \epsilon_{01} \\ \epsilon_{02} \\ \epsilon_{03} \end{bmatrix} - J_{12}J_{22}^{-1}\begin{bmatrix} \epsilon_{04} \\ \epsilon_{05} \\ \epsilon_{06} \end{bmatrix}\right\} \quad (37a)$$

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{bmatrix} \leftarrow \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{bmatrix} + \delta\dot{\boldsymbol{\lambda}} \quad (37b)$$

$$\begin{bmatrix} \lambda_2 \\ \lambda_3 \\ t_f \end{bmatrix} \leftarrow \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ t_f \end{bmatrix} + J_{22}^{-1}\left\{\begin{bmatrix} \epsilon_{04} \\ \epsilon_{05} \\ \epsilon_{06} \end{bmatrix} - J_{21}\delta\dot{\boldsymbol{\lambda}}\right\} \quad (37c)$$

The velocity-to-be-gained algorithm is employed 10 to 15 sec before cutoff. During this time only Eq. (37c) is iterated with $\delta\dot{\boldsymbol{\lambda}}$ set to zero.

The active guidance policy is the same whether Op-Ex is being used for ascent or for orbital maneuvers. As an example of Op-Ex versatility, the direct-ascent-to-rendezvous option can be directly applied, except for minor changes in initialization, to all intermediate orbital burns and their subsequent coast arcs, and the orbit injection option could be applied to the final burn.

Effect on Op-Ex of Measurement of Engine Performance during Active Guidance

Op-Ex guidance relies on propulsion performance descriptions for the various stages to extrapolate the trajectory forward to thrust termination. In active guidance, variation of engine performance from that specified will cause the extrapolated trajectory to be in error. However, because the guidance is "closed-loop" by design, (that is, the trajectory is recursively extrapolated every 10 sec during early stages and more frequently in the final stage) such performance variations do not seriously affect the satisfaction of guidance objectives. But the fuel optimizing ability of Op-Ex is handicapped.

A "mass-rate" prediction scheme was implemented for study purposes and was found to enhance the fuel efficiency of Op-Ex guidance in the presence of off-nominal engine performance. The amount of Δv saved, in every case, was a few feet per second.

The "mass-rate" predictor is formulated as follows: the theoretical Δv gained in stage j between the k and $(k + 1)$ predictions of Op-Ex is, if c^* is constant,

$$\Delta v = -c_j^* \log_e \left\{\frac{m_{0j} + \dot{m}_j'(t_{k+1} - t_{j-1})}{m_{0j} + \dot{m}_j'(t_k - t_{j-1})}\right\} \quad (38)$$

t_{j-1} is time of stage ignition, a measured quantity, and $\dot{m}_j'$ is estimated mass rate of the jth stage. The measured $\Delta v'$ gained is approximately

$$\Delta v' \simeq \mathbf{u}_k \cdot (\mathbf{w}_{k+1} - \mathbf{w}_k) \quad (39)$$

where $\mathbf{w}$ is the velocity-meter output vector. Predicted average mass rate is then

$$\dot{m}_j' = \frac{-m_{0j}\{1 - \exp(-\Delta v'/c_j^*)\}}{\{t_{k+1} - t_{j-1}\} - \{t_k - t_{j-1}\}\exp(-\Delta v'/c_j^*)} \quad (40)$$

Simulation Test Results

To evaluate its behavior, Op-Ex was implemented as a guidance mode in a general purpose vehicle simulation program for the digital computer. Data from three simulated trajectories will be presented. The three trajectories are variations of ascent, where the guidance objectives are orbit insertion, direct-ascent intercept, and direct-ascent rendezvous. In all cases, a spherical Earth model was assumed which had a gravitational coefficient of 1.40829×10^{16} ft^3/sec^2 and a radius of 2.090992×10^7 ft.

The ascent trajectories were initialized at conditions "outside" the atmosphere where Op-Ex is applicable. The initial conditions used are shown in Table 1.

Starting from these conditions, the first of two stages was ignited and burned to fuel depletion. A "fuel settling" phase of 15-sec duration was initiated after first stage burnout and continued until second-stage ignition. The second stage was ignited and burned until terminated on command from Op-Ex when guidance objectives were achieved. The rocket specifications used for ascent are shown in Table 2.

In all cases, a common, inertially fixed "target plane" was used for the specification of guidance objectives. The "target plane" was inclined 30° relative to the plane of initial motion, that is, the plane defined by the initial position and velocity vectors of the rocket. The intersection of the two planes lay 10° downrange from the initial position vector.

Case 1: The guidance objective of case 1 was to place the rocket into a circular orbit, of radius 2.1625×10^7 ft, lying in the target plane. This requires a "dog-leg" ascent trajectory which includes a plane change. To show the effect on fuel optimality and Op-Ex behavior, the maximum integration step-size used by the explicit-predictor integration algorithm was varied from 10 to 100 to 1000 sec. The time step of 1000 sec was limited by Eq. (8), and was, in effect, a requirement for one integration step per stage. The results of this variation are shown in Table 3.

Table 2 Booster rocket specifications

Stage	Init. wt., lb	Thrust, lb	Wt. flow rate, lb/sec	Burn time, sec
1	790,000	570,000	−1425	375
Fuel Settling	200,000	500	−2	15
2	199,970	110,000	−275	...

Table 3 The effect on Op-Ex behavior and fuel optimality of explicit-predictor integration step-size

Maximum integration time step, sec	10	100	1,000
Initial estimate of time of thrust termination, sec	645.92	646.61	653.53
Actual time of thrust termination, sec	645.909	645.910	645.967
Total Δv consumed, fps	20,104.51	20,104.54	20,106.28
Number of passes for initial convergence	5	5	6

Table 4 Comparison of intercept and rendezvous fuel requirements to identical target

	Direct ascent rendezvous	Direct ascent intercept
Initial burning time, sec	668.125	667.388
Initial Δv expended, fps	20,725.99	20,704.87
Δv required for circularization at intercept point, fps	797.43	907.99

During each flight, Op-Ex was reconverged every 10 sec during the first stage and every 5 sec during the second. Only one pass through the guidance algorithm was required for each reconvergence when the maximum integration step size was 10 or 100 sec. When the step size was 1000 sec (one step per stage), two passes were required to achieve convergence during the initial three-fourths of the first stage. Errors in the satisfaction of guidance objectives were trivial for all trajectories, and were representative of the numerical resolution of the simulation model.

To qualitatively evaluate the ability of Op-Ex to function in the presence of unexpected large variations in engine performance, the same orbit injection was tried, but with thrust and mass-flow rate in the second stage reduced by half. When the second stage was ignited, Op-Ex adapted to the situation, by means of its mass-rate prediction equations, by increasing the time of thrust termination and reoptimizing the steering policy to compensate for reduced engine performance. The number of passes required on each guidance reconvergence increased to four. Within 60 sec after stage two ignition, Op-Ex had increased its estimated time of thrust termination from 648.09 sec to 1046.75 sec. Op-Ex predicted termination at 1081.35 sec just 130 sec after stage two ignition, and had again decreased the number of passes required for each guidance reconvergence to one. The actual time of thrust termination was 1081.897 sec.

Case 2: The guidance objective of case 2 was to place the rocket on a trajectory which would intercept a specified point in the target plane at a specified time. The point in the target plane was defined to be 120 degrees from the intersection of the target and initial planes, and at a radius of 2.36×10^7 ft. The time of intercept was specified as 2260 sec after first stage ignition. The results from this case are presented in Table 4.

Case 3. The guidance objective of case 3 was to place the rocket on a rendezvous trajectory with the same target as specified in case 2. In addition, however, the target was specified to be in a circular orbit in the target plane. Fuel minimization in this case must include the required velocity increment for orbit circularization at the point of rendezvous. Results from this case are also shown in Table 4.

On examining Table 4, it can be seen that the optimization procedures worked just as expected. When the requirement was to intercept, the initial and only Δv required was less than that for rendezvous. However, when the requirement was to rendezvous, the sum of the initial Δv and the Δv required to circularize was less by 89.44 fps.

Summary

The difficulties in applying optimal control theory to rocket powered flight outside the atmosphere have been successfully overcome. The resulting algorithm has been shown to be both an effective and versatile guidance technique for use in real time and an accurate fuel requirements estimator for use in prelaunch or preignition flight planning. Its demonstrated convergence properties show that it is reasonably insensitive to the required initial guesses of the costate variables.

The computational burden of Op-Ex is well within the capabilities of nearly all contemporary computers used for space missions. Its versatility may well result in a net savings of the computing resourse, compared with the requirements of current guidance techniques, as the sophistication of space missions increases.

It should be noted that the Op-Ex algorithm has growth potential for the inclusion of additional constraints, such as a specified attitude at thrust termination. Implementation would require no more than a modification of the control policy; the procedures for trajectory extrapolation and iterative adjustment of parameters would be essentially unchanged.

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